Abstract homomorphisms of special unitary groups

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October 13, 2021

Outline

1. Borel-Tits conjecture
   - Statement of conjecture
   - Known cases
   - New result for SU_{2n}
2. Methods involved in the proof
   - Elementary subgroup and Steinberg group
   - Algebraic ring associated to an abstract representation
   - Lifting to the Steinberg group
   - Descending to a rational representation

Borel-Tits conjecture, original

Conjecture (BT, 1973)

Let G and G' be algebraic groups defined over infinite fields k and k', respectively. If $\rho\colon G(k) \to G'(k')$ is any abstract homomorphism such that $\rho(G^+)$ is Zariski-dense in $G'(k')$, then there exists a commutative finite-dimensional k'-algebra A and a ring homomorphism $f : k \to A$ such that $\rho = \sigma \circ r_{A/k'} \circ F$ with σ a morphism of algebraic groups.

$$\begin{array}{ccc} G(k) & \xrightarrow{\quad\rho\quad} & G'(k') \\ {\scriptstyle F}\searrow & & \nearrow{\scriptstyle \sigma} \\ G_A(A) & \xrightarrow[\cong]{r_{A/k'}} & R_{A/k'}(G_A)(k') \end{array}$$

$F : G(k) \to G_A(A)$ is induced by f, and $r_{A/k'} : G_A(A) \to R_{A/k'}(G_A)(k')$ is the canonical isomorphism.

Borel-Tits conjecture, restated

Conjecture (BT, 1973)

Let G and G' be algebraic groups defined over infinite fields k and k', respectively. If $\rho\colon G(k) \to G'(k')$ is any abstract homomorphism such that $\rho(G^+)$ is Zariski-dense in $G'(k')$, then there exists a commutative finite-dimensional k'-algebra A and a ring homomorphism $f : k \to A$ such that $\rho = \sigma \circ F$ with σ a morphism of algebraic groups.

$$\begin{array}{ccc} G(k) & \xrightarrow{\quad\rho\quad} & G'(k') \\ & {\scriptstyle F}\searrow \quad \nearrow{\scriptstyle \sigma} & \\ & G_A(A) & \end{array}$$

A factorization of this form is called a **standard description** of ρ.

Who	When	Case
Steinberg	1968	$G = G'$ Chevalley group with irreducible root system, k perfect, ρ abstract automorphism
Tits	1972	$k = k' = \mathbb{R}$
Borel, Tits	1973	G absolutely simple, simply-connected, k-isotropic, and G' absolutely simple, another version with G' reductive, technical
Weisfeiler	1982	$G \cong G'$ absolutely simple, simply-connected, anisotropic, split by quadratic extension, ρ abstract isomorphism
Seitz	1997	char $k > 0$ and k perfect
L. Lifschitz, A. Rapinchuk	2001	G absolutely simple simply-connected Chevalley group, char $k = 0$, $R_u(G')$ commutative
I. Rapinchuk	2011	$G(k)$ split, k' algebraically closed, char $k' = 0$
I. Rapinchuk	2013	$G = \mathrm{SL}_{n,D}$, $n \geq 3$, k' algebraically closed, char $k' = 0$, D f.d. central division algebra over char $= 0$

Description of $\mathrm{SU}_{2n}(L, h)$

Let L/k be a quadratic extension in char $= 0$. Let $\sigma \in \mathrm{Gal}(L/k)$ be the nontrivial element, denoted $\sigma(x) = \overline{x}$. For a k-algebra R, extend σ to $R_L = R \otimes_k L$ by acting on the L part.
Fix $n \geq 2$ and let h be a skew-hermitian form of maximal Witt index on L^{2n}. Let $G = \mathrm{SU}_{2n}(L, h)$ be the isometry group of h.
We can choose an L-basis of L^{2n} so that the matrix of h is

$$H = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

Then given a k-algebra R, the group of R-points of G is identified with

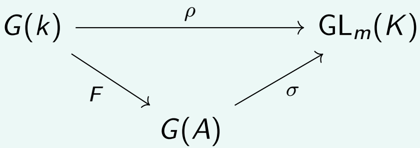
$$G(R) = \{X \in \mathrm{SL}_{2n}(R_L) : X^* H X = H\}$$

$X^* = \overline{X}^t$ is the conjugate transpose.

New result for $\mathrm{SU}_{2n}(L, h)$

Theorem (I. Rapinchuk and R, 2021)

Let $G = \mathrm{SU}_{2n}(L, h)$ *and let* K *be an algebraically closed field of characteristic zero. Given abstract homomorphism* $\rho : G(k) \to \mathrm{GL}_m(K)$*, set* $H = \overline{\rho(G(k))}$*. If* $R_u(H)$ *is commutative, then there exists a commutative finite-dimensional* K*-algebra* A*, a ring homomorphism* $f : k \to A$ *with Zariski-dense image, and a morphism of algebraic* K*-groups* $\sigma : G(A) \to H$ *such that* $\rho = \sigma \circ F$*, where* $F : G(k) \to G(A)$ *is the group homomorphism induced by* f*.*



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Elementary subgroups

Fix a commutative unital ring R.

Definition (Petrov and Stavrova)

Let G be a reductive algebraic group over R. Let $P \subset G$ be a parabolic subgroup with unipotent radical U_P and Levi subgroup L_P, and let P^- be the opposite parabolic subgroup (with respect to L_P). The **elementary subgroup** $E_P(R)$ is the subgroup of $G(R)$ generated (as an abstract group) by $U_P(R)$ and $U_{P^-}(R)$.

Theorem (Petrov and Stavrova, 2008)

Let G be a reductive algebraic group over R. Assume that for any maximal ideal $M \subset R$ all irreducible components of the relative root system of G_{R_M} have rank ≥ 2. Then $E_P(R)$ does not depend on the choice of a strictly proper parabolic subgroup P. In particular, $E(R) = E_P(R)$ is normal in $G(R)$.

Relative root subgroup embeddings

Let $G = \mathrm{SU}_{2n}(L, h)$, fix a maximal k-split torus S. Let $\Phi_k \subset X^*(S)$ be the relative root system (type C_n). For $\alpha \in \Phi_k$, there exists a vector k-group scheme V_α and a closed embedding of schemes $X_\alpha : V_\alpha \to G$, such that for any k-algebra R:

1. $E(R)$ is generated by the elements $X_\alpha(v)$ for all $\alpha \in \Phi_k$ and all $v \in V_\alpha(R)$.
2. $X_\alpha(v) \cdot X_\alpha(w) = X_\alpha(v + w)$ for all $v, w \in V_\alpha(R)$.
3. $s \cdot X_\alpha(v) \cdot s^{-1} = X_\alpha\Big(\alpha(s)v\Big)$ for all $s \in S(R), v \in V_\alpha(R)$.
4. (Chevalley commutator formula) For any $\alpha, \beta \in \Phi_k$ such that $\alpha \neq \pm\beta$, and for all $u \in V_\alpha(R), v \in V_\beta(R)$,

$$\Big[X_\alpha(u), X_\beta(v)\Big] = \prod_{\substack{i,j \geq 1 \\ i\alpha + j\beta \in \Phi_k}} X_{i\alpha + j\beta}\Big(N_{ij}^{\alpha\beta}(u, v)\Big)$$

for some polynomial maps $N_{ij}^{\alpha\beta} : V_\alpha(R) \times V_\beta(R) \to V_{i\alpha+j\beta}(R)$.

Concrete descriptions of V_α

Let $G_0 \subset G$ be a split subgroup of type C_n, so $G_0(k) \cong \mathrm{Sp}_{2n}(k)$.

For a long root α, the relative root space is one dimensional, and the root subgroups of G and G_0 coincide, so $V_\alpha = \mathbb{G}_a$.

For a short root α, the relative root space is two dimensional, and the associated root subgroup of $G_0(k)$ is the fixed subset of the Galois action of $\mathrm{Gal}(L/k)$ on the root subgroup $X_\alpha(V_\alpha(k)) \subset G(k)$, and $V_\alpha = \mathbb{G}_a^2$. In this case, identify $V_\alpha(k) \cong k \oplus k \cong L$.

The Steinberg group, 1

The **Steinberg group** $\widetilde{G}(R)$ is the abstract group generated by symbols $\widetilde{X}_\alpha(v)$ for $\alpha \in \Phi_k$ and $v \in V_\alpha(R)$, subject to relations (1) and (3) above.

1. $\widetilde{X}_\alpha(v) \cdot \widetilde{X}_\alpha(w) = \widetilde{X}_\alpha(v + w)$
2. $\left[\widetilde{X}_\alpha(u), \widetilde{X}_\beta(v)\right] = \prod_{\substack{i,j \geq 1 \\ i\alpha + j\beta \in \Phi_k}} \widetilde{X}_{i\alpha + j\beta}\left(N_{ij}^{\alpha\beta}(u, v)\right)$

The **natural projection** is $\pi_R : \widetilde{G}(R) \to E(R), \widetilde{X}_\alpha(v) \mapsto X_\alpha(v)$.

The Steinberg group, 2

Theorem (Stavrova, 2020)

For G an isotropic simply connected reductive group over a local ring R, if all irreducible components of Φ_P have rank ≥ 2, then $\ker \pi_R$ is central in $\mathrm{St}_P(R) = \widetilde{G}(R)$.

Corollary (I. Rapinchuk and R, 2021)

For $G = \mathrm{SU}_{2n}(L, h)$ and R a product of local k-algebras, $\ker \pi_R$ is central in $\widetilde{G}(R)$.

Sketch of proof: $\widetilde{G}(-)$ commutes with finite products (a little bit of work) and $E(-)$ commutes with finite products, then apply theorem.

The algebraic ring associated to an abstract homomorphism, 1

R - commutative unital ring.
K - algebraically closed field of characteristic zero.
$e_{ij}(r)$ - elementary matrix, differs from identity by having r in the ij position.
$\mathrm{E}_n(R)$ - elementary subgroup of $\mathrm{SL}_n(R)$, the subgroup generated by the $e_{ij}(r)$.

Kassabov-Sapir construction: Let $\rho : \mathrm{E}_n(R) \to \mathrm{GL}_m(K)$ be an abstract homomorphism ($n \geq 3$). Then $A = \overline{\rho(e_{13}(R))}$ has the structure of an algebraic ring, such that $f : R \to A, r \mapsto \rho(e_{13}(r))$ is a ring homomorphism. Addition in A is matrix multiplication, making use of $e_{13}(r) \cdot e_{13}(s) = e_{13}(r+s)$.

The algebraic ring associated to an abstract homomorphism, 2

Φ - reduced irreducible root system of rank ≥ 2.
G - universal Chevalley group associated to Φ.
e_α - for $\alpha \in \Phi$, the root subgroup embedding $R \hookrightarrow G(R)$.
(Φ, R) is a **nice pair** if $2 \in R^\times$ when Φ contains a copy of B_2 and $2, 3 \in R^\times$ when $\Phi = G_2$.

Generalization by I. Rapinchuk: Assume (Φ, R) is a nice pair. Let $\rho : G(R)^+ \to \mathrm{GL}_m(K)$ be an abstract homomorphism. There is $\alpha \in \Phi$ such that $A_\alpha = \overline{\rho(e_\alpha(R))}$ has the structure of an algebraic ring, such that $f_\alpha : R \to A_\alpha, r \mapsto \rho(e_\alpha(r))$ is a ring homomorphism. For any $\alpha, \beta \in \Phi$, $A_\alpha \cong A_\beta$ as varieties, so identify them all with a single algebraic ring A.
For $\alpha \in \Phi$, there is an injective regular map $\psi_\alpha : A \to H = \overline{\rho(G(R)^+)}$ such that TFDC.

$$\begin{array}{ccc} R & \xrightarrow{e_\alpha} & G(R)^+ \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \rho} \\ A & \xrightarrow{\psi_\alpha} & H \end{array}$$

The algebraic ring associated to an abstract homomorphism, 3

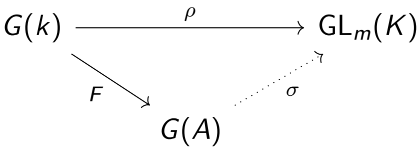
New extension: Let $G = \mathrm{SU}_{2n}(L, h)$ and let $\rho : G(k) \to \mathrm{GL}_m(K)$ be an abstract homomorphism. There is $\alpha \in \Phi_k$ such that $A_\alpha = \overline{\rho(X_\alpha(k))}$ has the structure of an algebraic ring, such that $f_\alpha : k \to A_\alpha, r \mapsto \rho(X_\alpha(r))$ is a ring homomorphism. For $\alpha, \beta \in \Phi_k$, $A_\alpha \cong A_\beta$ as varieties, so identify them all with a single algebraic ring A. For $\alpha \in \Phi_k$, there is a regular map $\psi_\alpha : V_\alpha(A) \to H = \overline{\rho(G(k))}$ such that TFDC.

$$\begin{array}{ccc} V_\alpha(k) & \xrightarrow{X_\alpha} & G(R)^+ \\ {\scriptstyle V_\alpha(f)}\downarrow & & \downarrow{\scriptstyle \rho} \\ V_\alpha(A) & \xrightarrow{\psi_\alpha} & H \end{array}$$

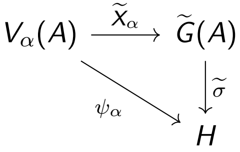
For long roots, $V_\alpha(k) = k$ and $V_\alpha(f) = f$ so this is just the same diagram as for the split subgroup $G_0 \cong \mathrm{Sp}_{2n}$. But for short roots $V_\alpha(k) = L$ so this is new.

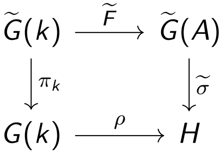
Lifting to the Steinberg group

Recall, the goal is to construct A, σ, and $f : k \to A$ inducing $F : G(k) \to G(A)$ making this commute.



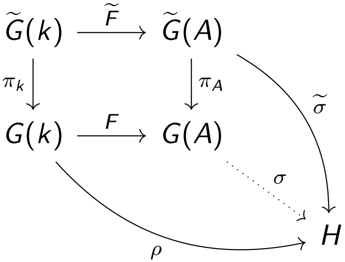
Strategy: write down what a lift of σ to $\widetilde{G}(A)$ should be, then descend to $G(A)$.
Define $\widetilde{\sigma} : \widetilde{G}(A) \to H$ by $\widetilde{\sigma}\left(\widetilde{X}_\alpha(v)\right) = \psi_\alpha(v)$. Then TFDC.





Descending to a rational representation, 1

The solid arrows make a commutative diagram. To complete the proof we need a k-rational morphism σ completing this diagram.



Descending to a rational representation, 2

Lemma

$\widetilde{\sigma} : \widetilde{G}(A) \to H$ is surjective, and H is connected and perfect.

Set $\overline{H} = H/Z(H)$ and $\nu : H \to \overline{H}$ the quotient map.

Proposition

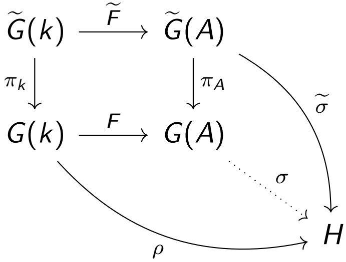
There exists a group homomorphism $\overline{\sigma} : G(A) \to \overline{H}$ such that $\overline{\sigma} \circ \pi_A = \nu \circ \widetilde{\sigma}$.

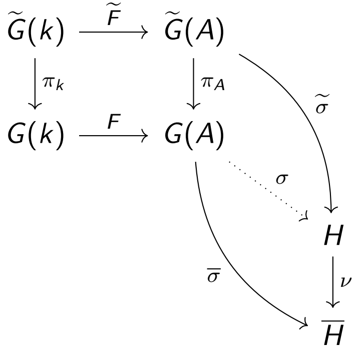
Proof.

Because G is quasi-split and A is a product of local K-algebras, $G(A) = E(A)$, so $E(A) \cong \widetilde{G}(A)/\ker \pi_A$. By previous lemma $\widetilde{\sigma}$ is surjective and $\ker \pi_A$ is central in $\widetilde{G}(A)$, so $\widetilde{\sigma}(\ker \pi_A) \subset Z(H)$, so $\widetilde{\sigma}$ induces $\overline{\sigma} : G(A) \to \overline{H}$ with the required property. □

Descending to a rational representation, 3

The solid arrows commute. Next: show $\overline{\sigma}$ is algebraic.





Sketch of proof: Write $G(A)$ as a product of root groups in some fixed order, and use some technical algebro-geometric lemmas from [Rap2013].
Next: lift $\overline{\sigma}$ to obtain σ.

Descending to a rational representation, 4

Given an ideal $I \subset A$, we have the congruence kernel $G(A, I) = \ker \Big(G(A) \to G(A/I) \Big)$.

Lemma

Let $J \subset A$ be the Jacobson radical. Then $G(A, J)$ is nilpotent and there is a Levi decomposition $G(A) = G(A, J) \ltimes G(\overline{A})$, where $\overline{A} \subset A$ is a semisimple subalgebra such that $A = \overline{A} \oplus J$ as K-vector spaces and $\overline{A} \cong A/J$ as K-algebras.

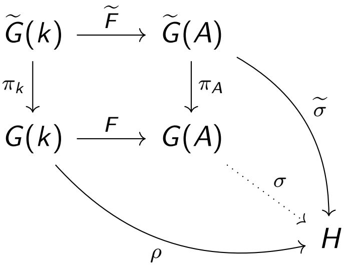
Note. A is a finite-dimensional K-algebra $\implies$ A is Artinian $\implies$ $J^d = 0$ for some $d \geq 1$ $\implies$ $\overline{A}$ with these properties exists by Wedderburn-Malcev Theorem.

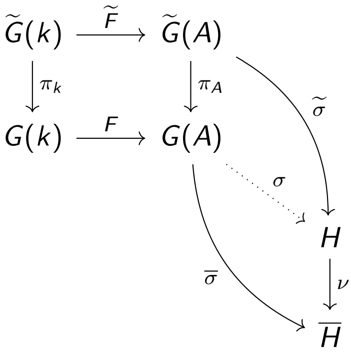
Lemma

If the unipotent radical $R_u(H)$ is commutative and $\operatorname{char} K = 0$, then $Z(H) \cap R_u(H) = \{e\}$, and $Z(H)$ is finite and contained in any Levi subgroup of H.

Descending to a rational representation, 5

Next step: lift $\overline{\sigma}$ to obtain σ.





Descending to a rational representation, 6

Theorem

Assuming $R_u(H)$ is commutative and char $K = 0$*, there exists a morphism of algebraic groups* $\sigma : G(A) \to H$ *such that* $\nu \circ \sigma = \overline{\sigma}$*, and this* σ *makes the previous diagram commute.*

Proof.

Recall the Levi decomposition $G(A) = G(A, J) \ltimes G(\overline{A})$. Then we have Levi decompositions $H = U \ltimes S$ and $\overline{H} = \overline{U} \ltimes \overline{S}$ where

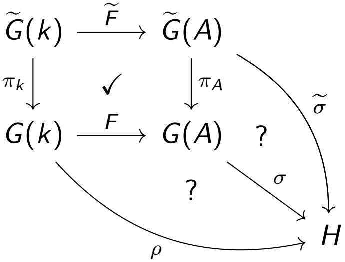
$$\overline{U} = \overline{\sigma}(G(A, J)) \qquad \overline{S} = \overline{\sigma}(G(\overline{A})) \qquad U = R_u(H) \qquad S = (\nu^{-1}(\overline{S}))^\circ$$

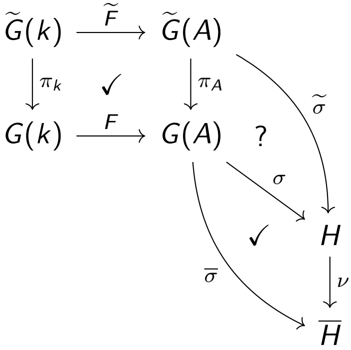
By previous lemma, $Z(H) \subset S$ so $\overline{S} = S/Z(H)$ and $\nu|_U : U \to \overline{U}$ is an isomorphism. Because $\nu : H \to \overline{H}$ is a central isogeny and $G(\overline{A})$ is simply connected, there exists a lift $\sigma_S : G(\overline{A}) \to S$ such that $\nu|_S \circ \sigma_S = \overline{\sigma}|_{G(\overline{A})}$.

Set $\sigma_U = \nu|_U^{-1} \circ (\overline{\sigma}|_{G(A,J)})$, then $\sigma = (\sigma_U, \sigma_S) : G(A) \to H$ is a morphism of algebraic groups and $\nu \circ \sigma = \overline{\sigma}$.

Descending to a rational representation, 7

Now we have σ. Next we check that the upper triangle commutes.

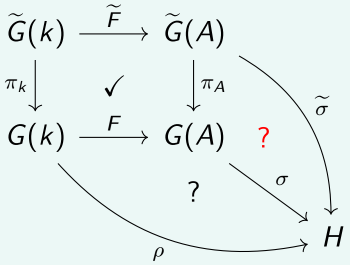




Descending to a rational representation, 8a

Proof (continued).

Define $\chi : \widetilde{G}(A) \to H$ by $g \mapsto \widetilde{\sigma}(g)^{-1} \cdot (\sigma \circ \pi_A)(g)$.

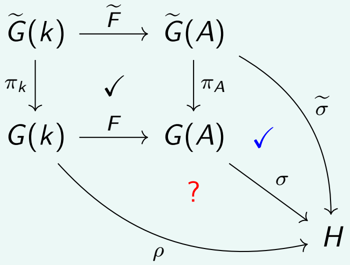


□

Descending to a rational representation, 8b

Proof (continued).

Define $\chi : \widetilde{G}(A) \to H$ by $g \mapsto \widetilde{\sigma}(g)^{-1} \cdot (\sigma \circ \pi_A)(g)$. We have $\nu \circ \widetilde{\sigma} = \overline{\sigma} \circ \pi_A = \nu \circ \sigma \circ \pi_A$. So the image of χ is contained in $\ker \nu = Z(H)$. Since $\widetilde{G}(A)$ is perfect, χ must be trivial, so $\sigma \circ \pi_A = \widetilde{\sigma}$.

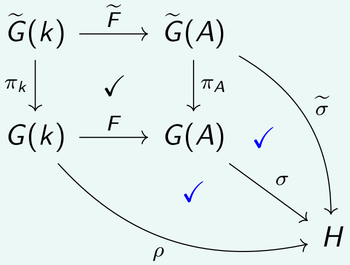


□

Descending to a rational representation, 8c

Proof (continued).

Define $\chi : \widetilde{G}(A) \to H$ by $g \mapsto \widetilde{\sigma}(g)^{-1} \cdot (\sigma \circ \pi_A)(g)$. We have $\nu \circ \widetilde{\sigma} = \overline{\sigma} \circ \pi_A = \nu \circ \sigma \circ \pi_A$. So the image of χ is contained in $\ker \nu = Z(H)$. Since $\widetilde{G}(A)$ is perfect, χ must be trivial, so $\sigma \circ \pi_A = \widetilde{\sigma}$. Since π_k is surjective, that commutativity of the remaining triangle is a formality.



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New result for $\mathrm{SU}_{2n}(L, h)$

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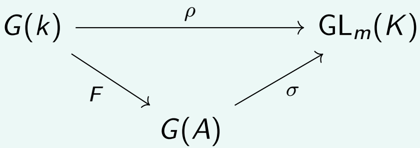

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